Tower decomposition of Hilbert class fields

Jared Guissmo E. Asuncion

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We discuss the solution given in Enge (2003) more thoroughly.

Let $K = \mathbb{Q}(\sqrt{-D})$. Let $\mathcal{O}_K$ be its ring of integers. Let

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Definition

The Weber polynomial is a defining polynomial for the Hilbert class field H_K given by

$$\mathscr{W}_D(X) = \prod_{\mathfrak{k} \in \mathrm{Cl}(\mathcal{O}_K)} (X - x_{\mathfrak{k}})$$

where $x_{\mathfrak{k}}$ is a complex number dependent on $\mathfrak{k}$.

Example

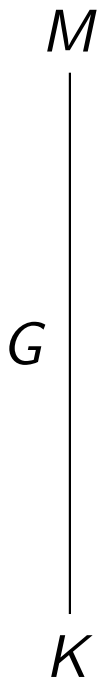
Let $K = \mathbb{Q}(\sqrt{-D})$ *where* $D = 335$. *Let* $\mathfrak{z}$ *be the trivial class group of* $\mathrm{Cl}(\mathcal{O}_K)$. *We have*

$$H_K = K(x_{\mathfrak{z}})$$

with defining polynomial

$$\begin{aligned}\mathscr{W}_D(X) = {} & X^{18} - 4X^{17} - 20X^{16} - 55X^{15} - 106X^{14} - 144X^{13} \\ & - 163X^{12} - 174X^{11} - 179X^{10} - 171X^9 - 144X^8 - 102X^7 \\ & - 64X^6 - 42X^5 - 33X^4 - 25X^3 - 14X^2 - 5X - 1.\end{aligned}$$

Let M be a Galois extension of a field K with $G = \mathrm{Gal}(M/K)$.
Let x be a root of f such that $M = K(x)$.



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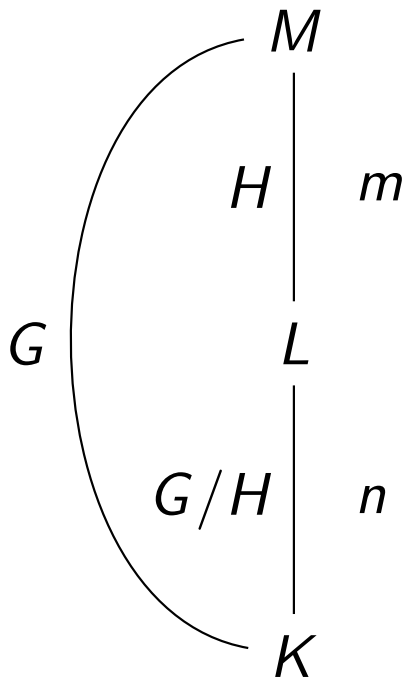

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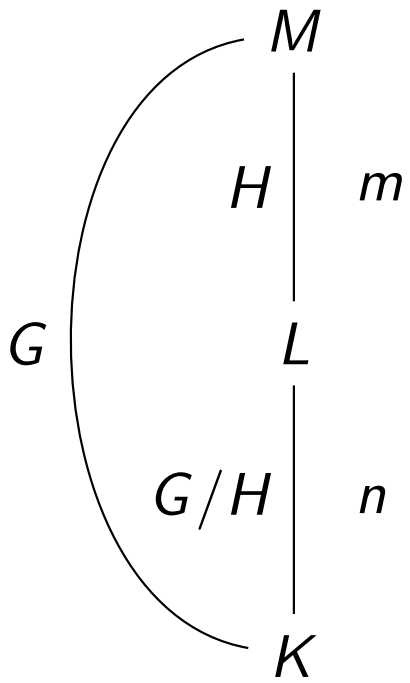

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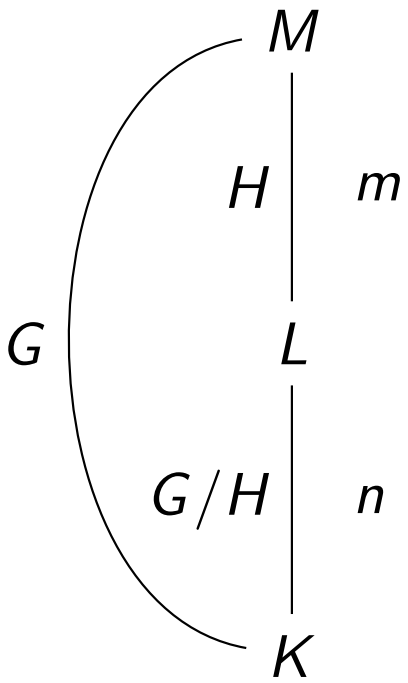

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$V = ???$

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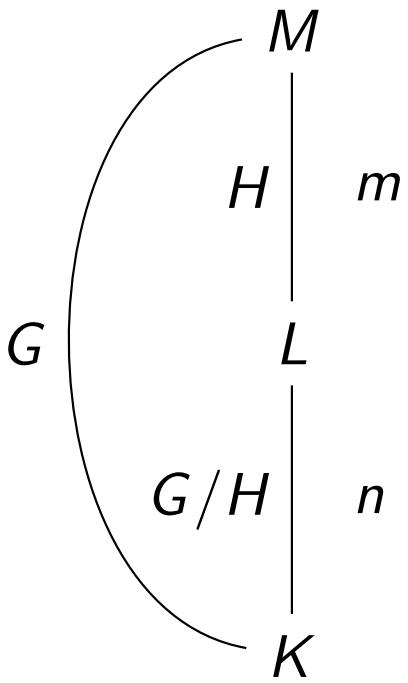

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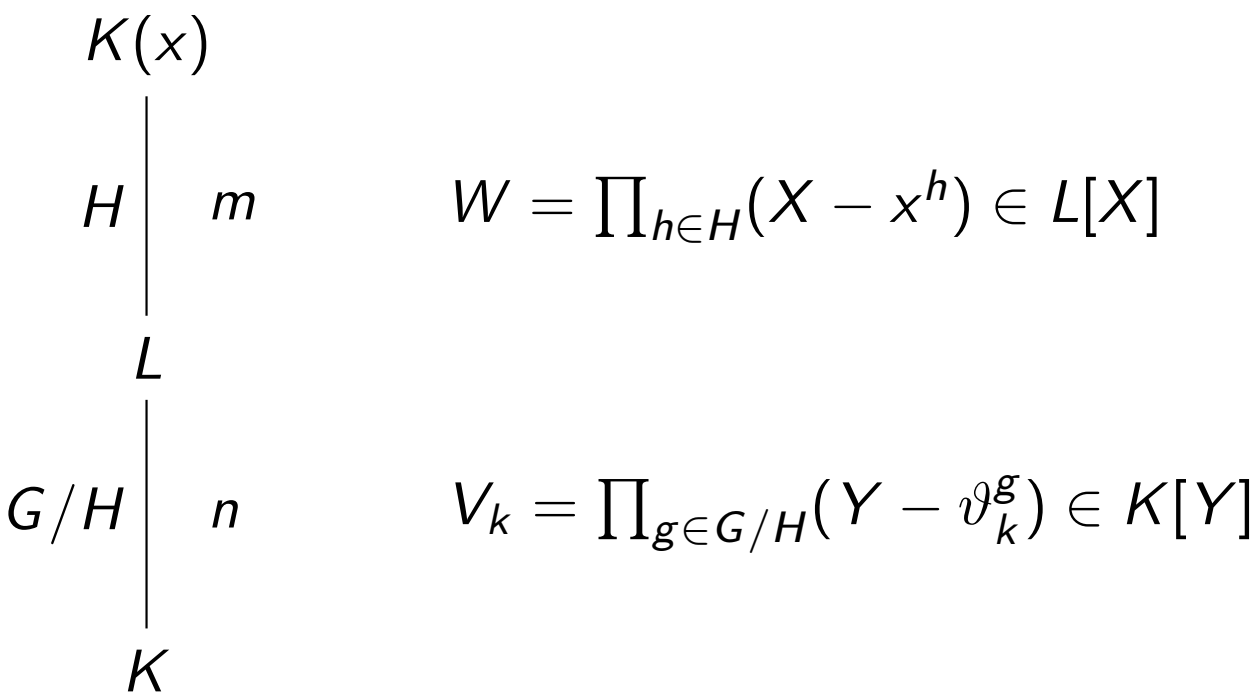

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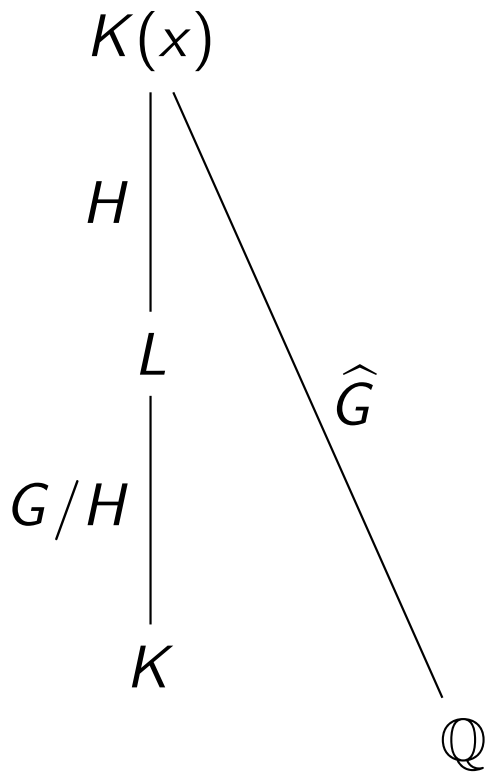

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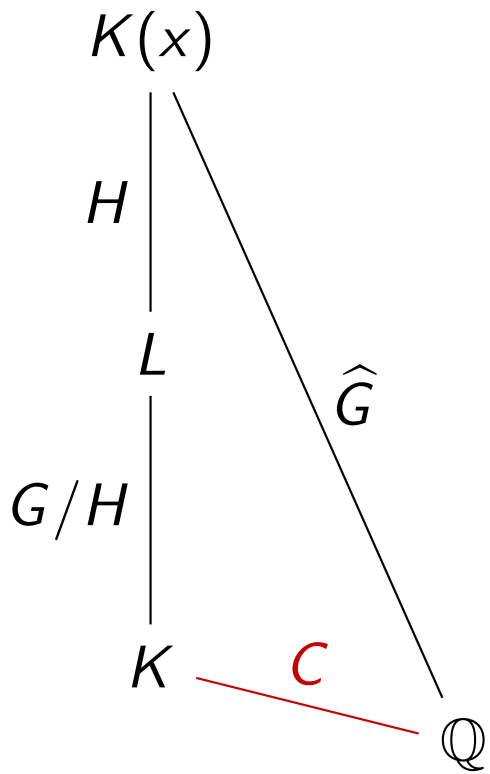

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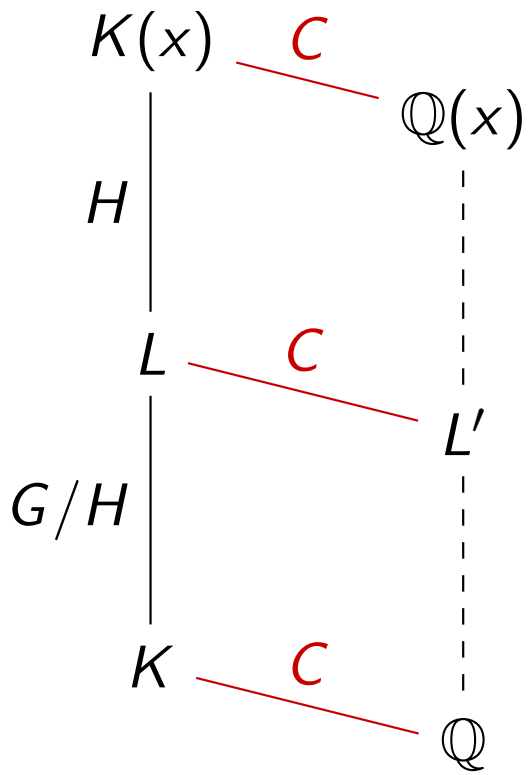
We have corresponding smaller fields

$$K' = \mathbb{Q} \qquad M' = \mathbb{Q}(x)$$



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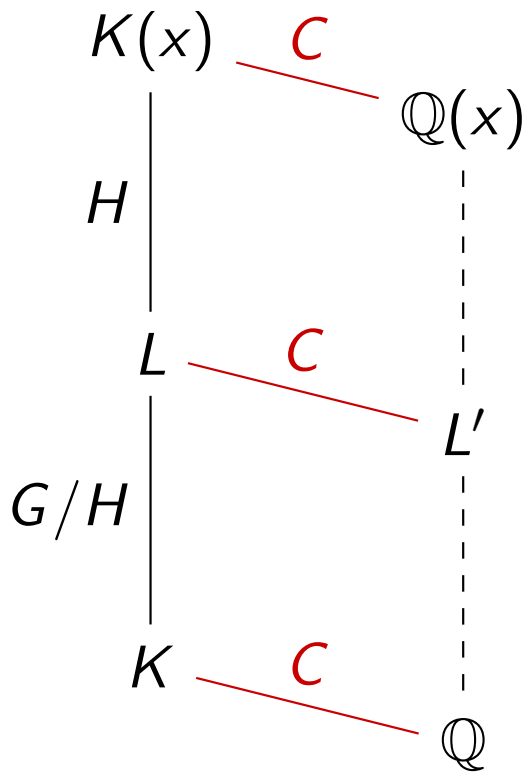

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Example

Let $D = 335$

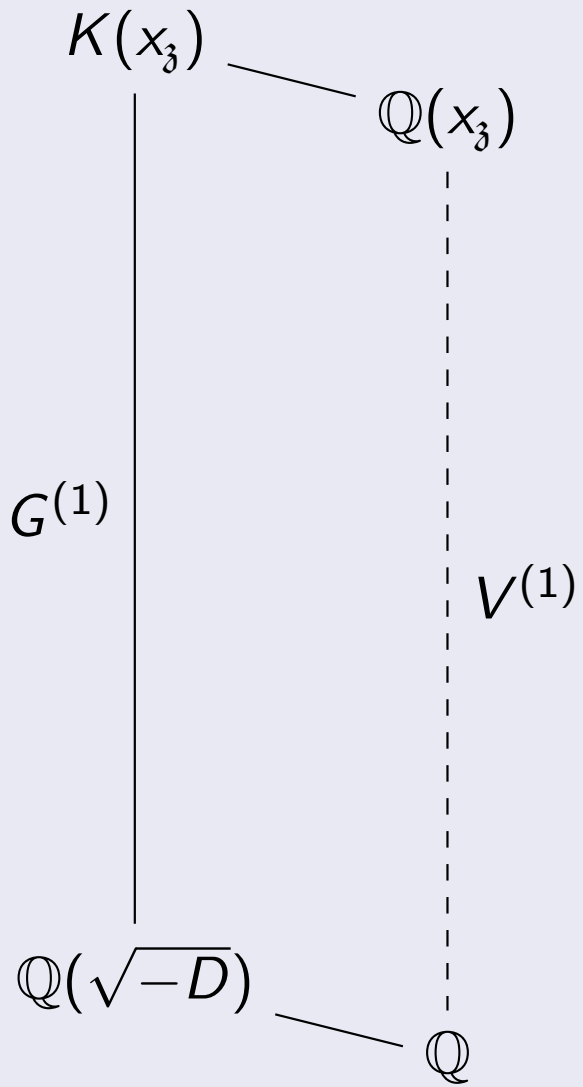
$$K = \mathbb{Q}(\sqrt{-335})$$
$$M = H_K = K(x_{\mathfrak{z}})$$

where $\mathfrak{z} \in \mathrm{Cl}(\mathcal{O}_K)$ *is the trivial ideal class.*

Note that

$$G^{(1)} = G \cong \mathrm{Cl}(\mathcal{O}_K) = \langle \mathfrak{k} \rangle$$

is cyclic with order 18.



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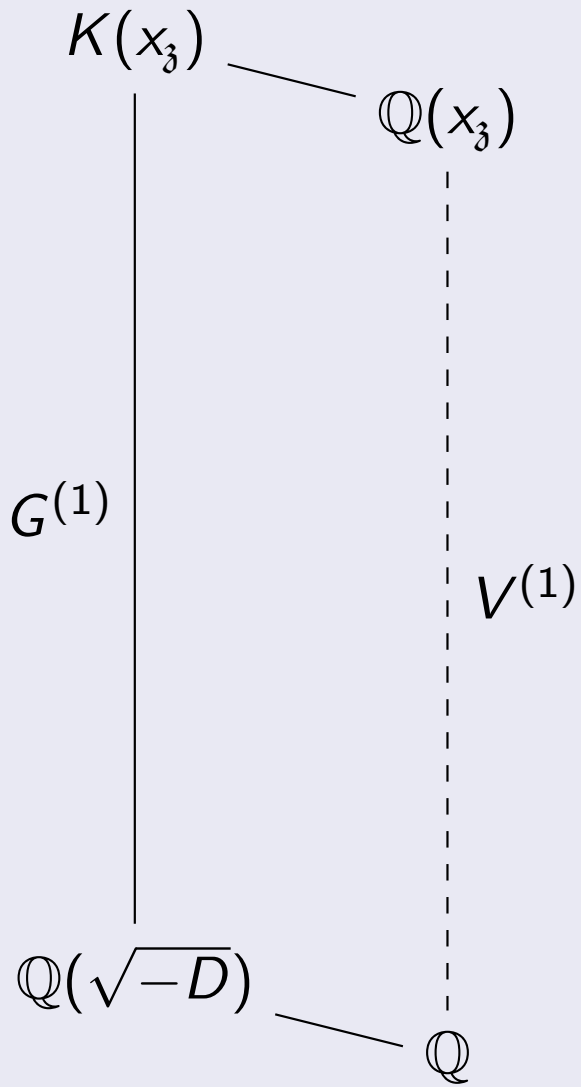
$$G^{(1)} := \langle \mathfrak{k} \rangle$$
$$H^{(1)} := \langle \mathfrak{k}^6 \rangle$$
$$V^{(1)} := \mathscr{W}_{335}$$



Example

$$G^{(2)} := \langle \mathfrak{t} \rangle / \langle \mathfrak{t}^6 \rangle$$
$$H^{(1)} := \langle \mathfrak{t}^6 \rangle$$
$$W_1 :\approx X^3 - 6.9X^2 - 5.7X - 5.1$$
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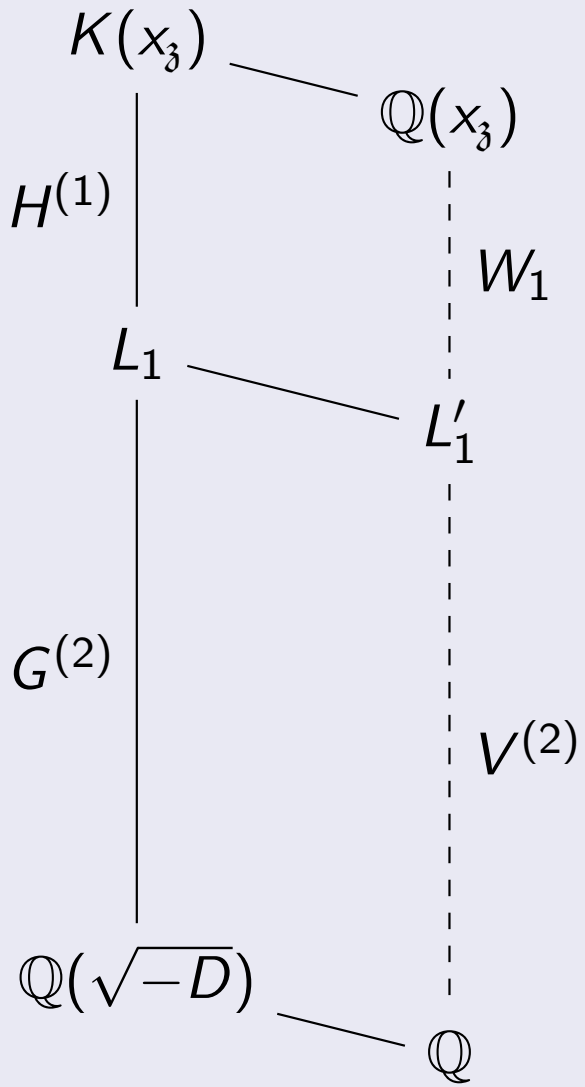

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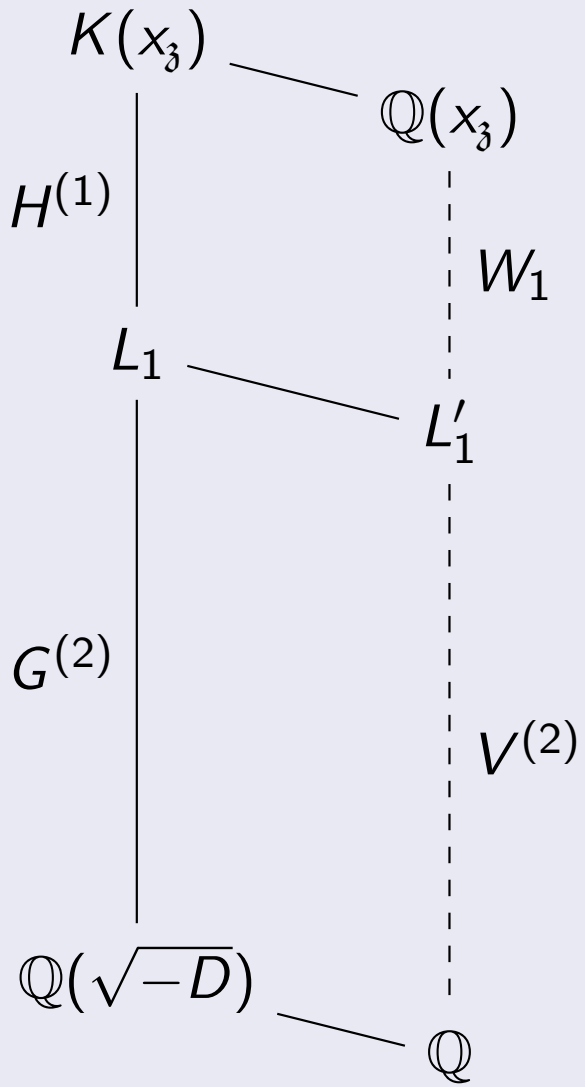

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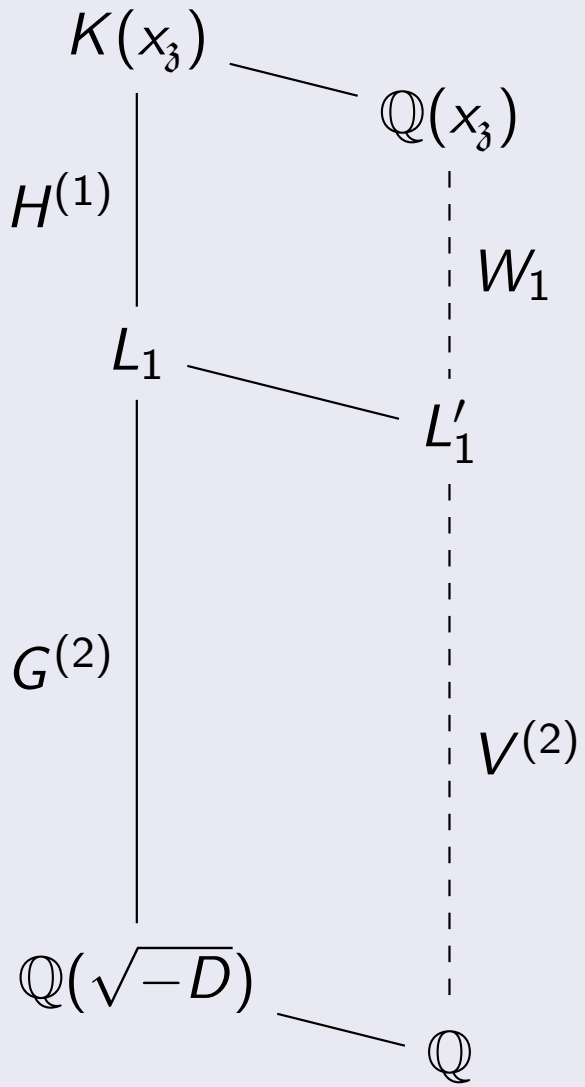

Problem

Coefficients of W are floating point numbers. We want to express them symbolically.

Lemma (Hecke representation)

Let $L = K(\alpha)$ and $V \in K[Y]$ a minimal polynomial for α. We can write any $\vartheta \in L$ as

$$\vartheta = \frac{g_\vartheta(\alpha)}{V'(\alpha)}$$

with

$$g_\vartheta(Y) = \sum_{i=0}^{n-1} \vartheta_i \frac{V(Y)}{Y - \alpha_i} \in K[Y].$$

We can solve the following Hecke representations for the coefficients of

$$W_1 \approx X^3 - 6.938X^2 - 5.742X - 5.124.$$

We have

$$\begin{aligned} V^{(2)'}(Y) &= 6Y^5 + 20Y^4 - 76Y^3 + 24Y^2 - 22Y - 1 \\ g_{1,2}(Y) &= -4Y^5 + 38Y^4 - 24Y^3 + 44Y^2 + 5Y + 6 \\ g_{1,1}(Y) &= -Y^5 + 43Y^4 - 52Y^3 + 26Y^2 - 21Y - 7 \\ g_{1,0}(Y) &= -4Y^5 + 23Y^4 - 9Y^3 - 11Y^2 + 10Y - 4 \end{aligned}$$

Hence

$$-6.938 \leftrightarrow \frac{g_{1,2}(Y)}{V^{(2)'}(Y)} \qquad -5.742 \leftrightarrow \frac{g_{1,1}(Y)}{V^{(2)'}(Y)} \qquad -5.124 \leftrightarrow \frac{g_{1,0}(Y)}{V^{(2)'}(Y)}$$

Example

$$W_1 := X^3 + \frac{1}{V^{(2)\prime}(Y)} \sum_{j=0}^{2} g_{1,j}(Y) X^j$$

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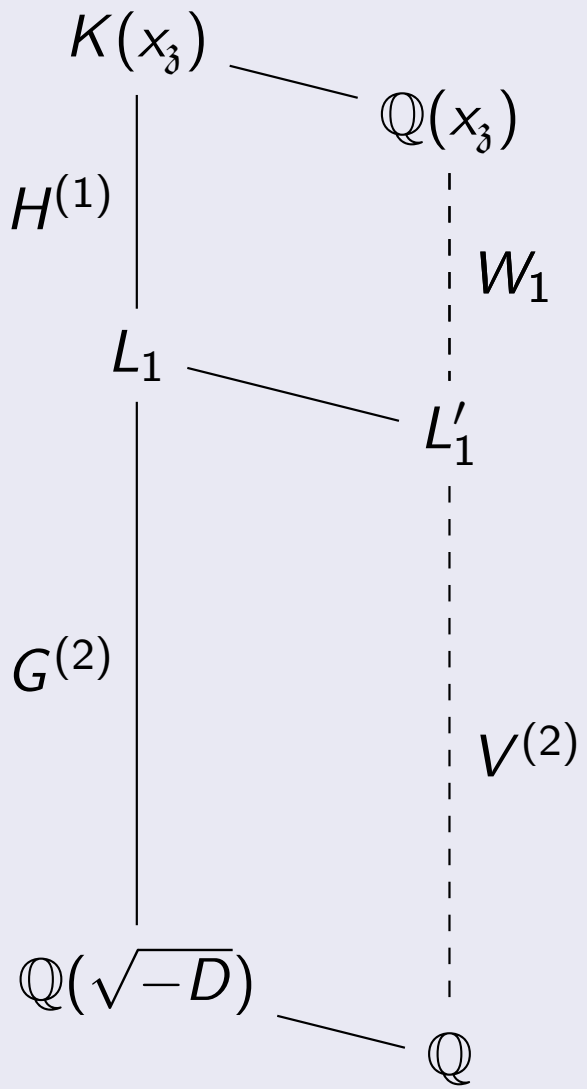

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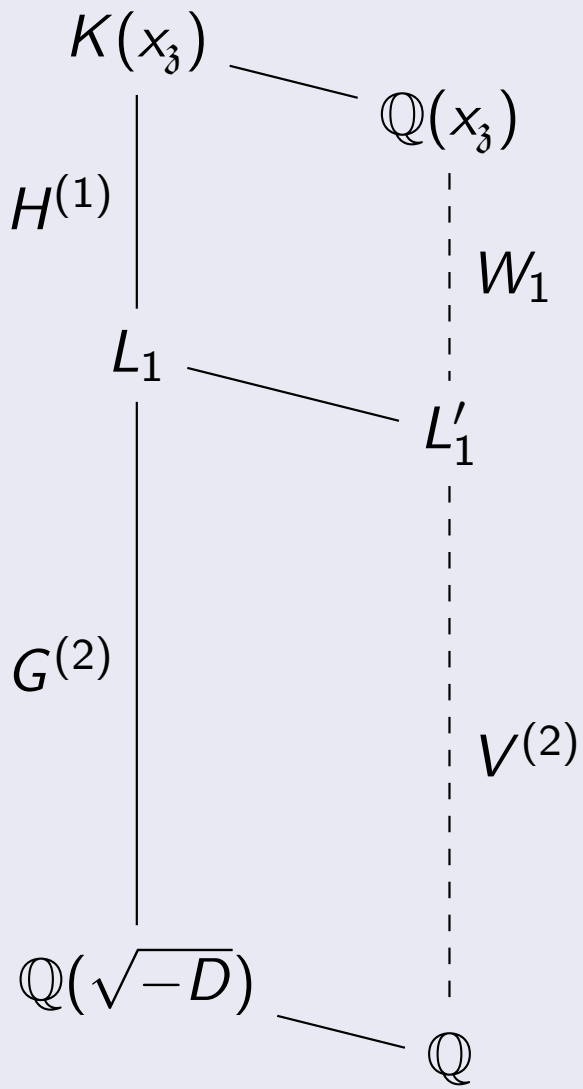

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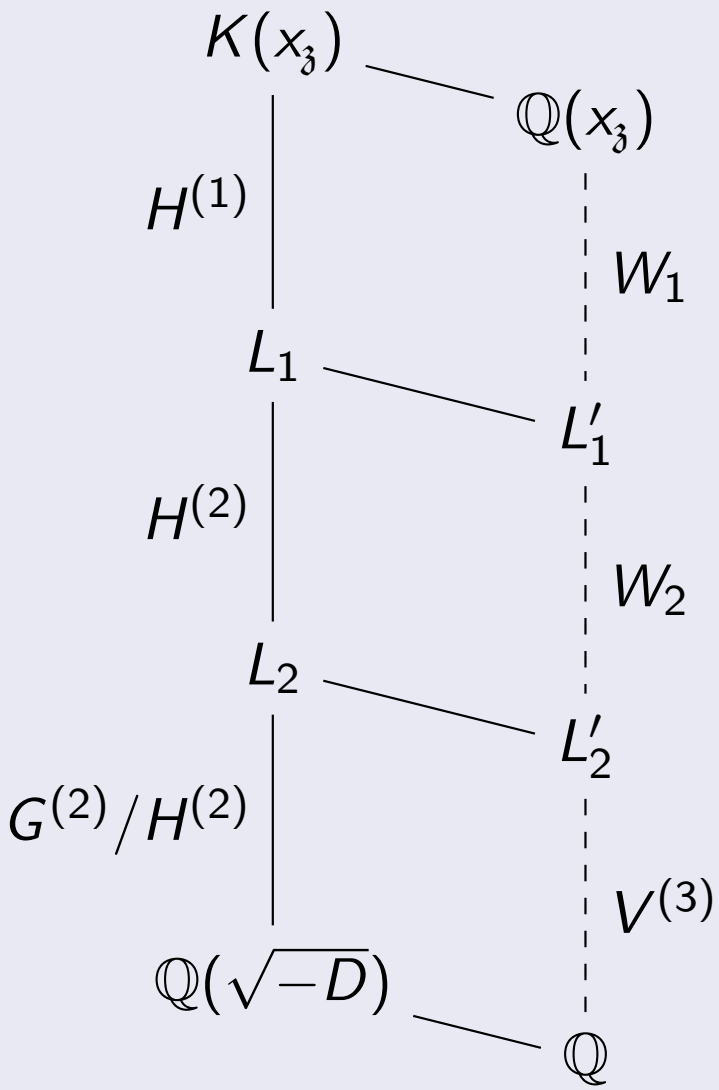
$$V^{(3)} := Z^2 - 4Z - 16$$



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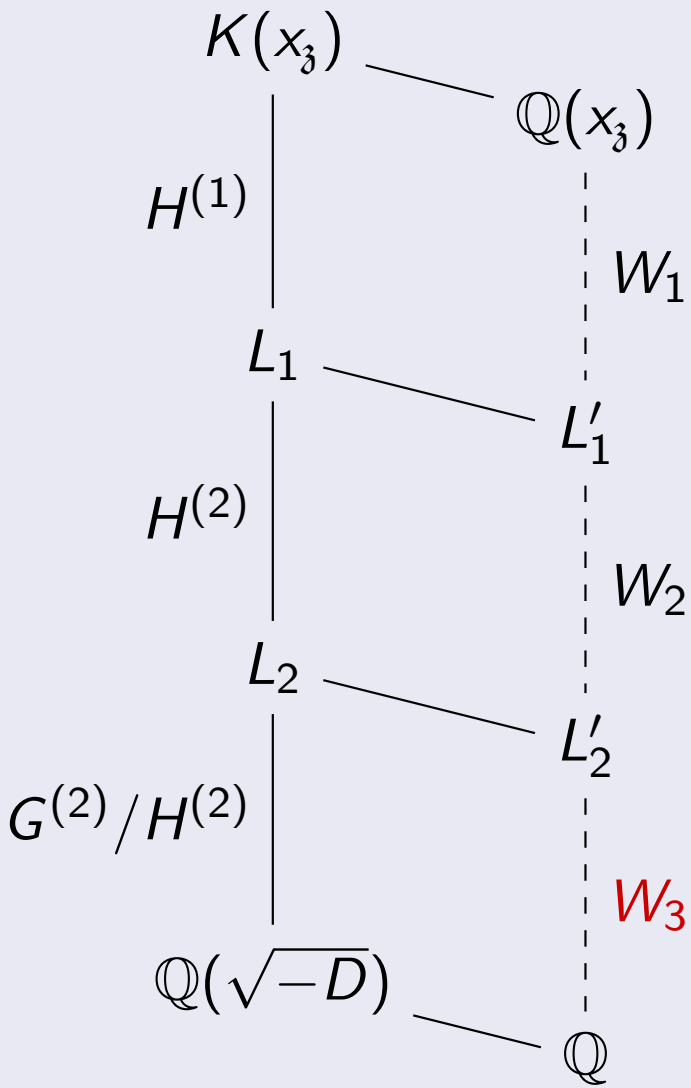

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We want an ordering of the roots

$$x_0, \ldots, x_{mn-1}$$

such that

- Each block of m roots will correspond to the elements of one of the cosets gH of G/H.
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This way, we can set

$$U_i(X) := \prod_{j=0}^{m-1} (X - x_{im+j}).$$

For the next iteration, we have a new set of roots

$$\vartheta_0, \ldots, \vartheta_{m-1}.$$

We want the first $m^{(2)} = |H^{(2)}|$ elements of this to be from $H^{(2)}$. And so on until the last iteration.

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Definition

A normal series of a group G is a sequence of normal subgroups of G such that

$$H_0 := 1 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_t := G.$$

Moreover, each H_i/H_{i-1} is called a factor group.

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$$H^{(\iota)} = \frac{H_\iota}{H_{\iota-1}}$$

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In our example, we had

$$H^{(1)} = \langle \mathfrak{k}^6 \rangle \quad \text{and} \quad H^{(2)} = \langle \mathfrak{k}^2 \rangle / \langle \mathfrak{k}^6 \rangle$$

In our example, we (implicitly) had

$$1 \lhd \langle \mathfrak{k}^6 \rangle \lhd \langle \mathfrak{k}^2 \rangle \lhd G.$$

Start with the element of H_0.

$$\mathfrak{k}^0.$$

List the coset representatives of $H^{(1)} = H_1/H_0$.

$$\mathfrak{k}^0, \mathfrak{k}^6, \mathfrak{k}^{12}.$$

Multiply them to the current list in that order.

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$$1 \triangleleft \langle \mathfrak{k}^6 \rangle \triangleleft \langle \mathfrak{k}^2 \rangle \triangleleft G.$$

We continue from the previous list

$$\mathfrak{k}^0, \mathfrak{k}^6, \mathfrak{k}^{12}.$$

List the coset representatives of $H^{(2)} = H_2/H_1$.

$$\mathfrak{k}^0, \mathfrak{k}^2, \mathfrak{k}^4.$$

Multiply them to the current list in that order.

$$\mathfrak{k}^0, \mathfrak{k}^6, \mathfrak{k}^{12}, \mathfrak{k}^2, \mathfrak{k}^8, \mathfrak{k}^{14}, \mathfrak{k}^4, \mathfrak{k}^{10}, \mathfrak{k}^{16}.$$

In our example, we had

$$H^{(1)} = \langle \mathfrak{k}^6 \rangle \quad \text{and} \quad H^{(2)} = \langle \mathfrak{k}^2 \rangle / \langle \mathfrak{k}^6 \rangle$$

In our example, we (implicitly) had

$$1 \triangleleft \langle \mathfrak{k}^6 \rangle \triangleleft \langle \mathfrak{k}^2 \rangle \triangleleft G.$$

We continue from the previous list

$$\mathfrak{k}^0, \mathfrak{k}^6, \mathfrak{k}^{12}, \mathfrak{k}^2, \mathfrak{k}^8, \mathfrak{k}^{14}, \mathfrak{k}^4, \mathfrak{k}^{10}, \mathfrak{k}^{16}$$

List the coset representatives of $H^{(3)} = H_3/H_2$.

$$\mathfrak{k}^0, \mathfrak{k}^1$$

Multiply them to the current list in that order.

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Finally, we have the final list.

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Take the corresponding $x_{\mathfrak{k}}$. (Write $x_i = x_{\mathfrak{k}^i}$.)

$$x_0, x_6, x_{12}, x_2, x_8, x_{14}, x_4, x_{10}, x_{16}, x_1, x_7, x_{13}, x_3, x_9, x_{15}, x_5, x_{11}, x_{17}$$

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Using PARI, we have this list of roots $\mathcal{R}_V$.

x_0	7.8	x_1	$-0.27 + 1.5i$
x_6	$-0.41 + 0.70i$	x_7	$-0.26 - 0.48i$
x_{12}	$-0.41 - 0.70i$	x_{13}	$0.62 - 0.78i$
x_2	$-0.052 + 0.92i$	x_3	$-0.75 - 0.41i$
x_8	$-0.78 + 0.27i$	x_9	-1.1
x_{14}	$0.60 - 0.44i$	x_{15}	$-0.75 + 0.41i$
x_4	$0.60 + 0.44i$	x_5	$0.62 + 0.78i$
x_{10}	$-0.78 - 0.27i$	x_{11}	$-0.26 + 0.48i$
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Each partition represents the roots of one of the U^g.

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Each partition represents the roots of one of the U^g.
In particular,

$$U_0 = (x - x_0)(X - x_1)(X - x_2) = X^3 - 6.9X^2 - 5.7X - 5.1$$

Algorithm

INPUT:

- *An ordered list $\mathcal{L}$ of the elements of the group $G = \mathrm{Gal}(M/K)$, using the normal series*

$$\langle e \rangle = H_0 \lhd H_1 \lhd \cdots \lhd H_{t-1} \lhd H_t = G.$$

of G. Note that $|G| = d$.

- *An ordered list $\mathcal{R}_V = (x_0, \ldots, x_{d-1})$ of the roots of V obtained from taking the corresponding roots of $\mathcal{L}$.*

Algorithm

OUTPUT:

- *A list of intermediate fields L'_{i-1}/L'_i for $i = 1, \ldots, t$ (where $M_0 = L'_0$ and $K_0 = L'_t$) and their respective irreducible polynomials $W_i \in L'_i[X]$.*

Algorithm

ALGORITHM

For each $\iota = 1, 2, \ldots, t-1$, do the following:

1. *Compute the U_i's for this iteration.*
2. *Find $V^{(\iota+1)}$ by taking different V_k's and see if they're irreducible.*
3. *Express the coefficients of $W = U_0$ as their Hecke representations.*

Problem

How to multiply polynomials quickly?

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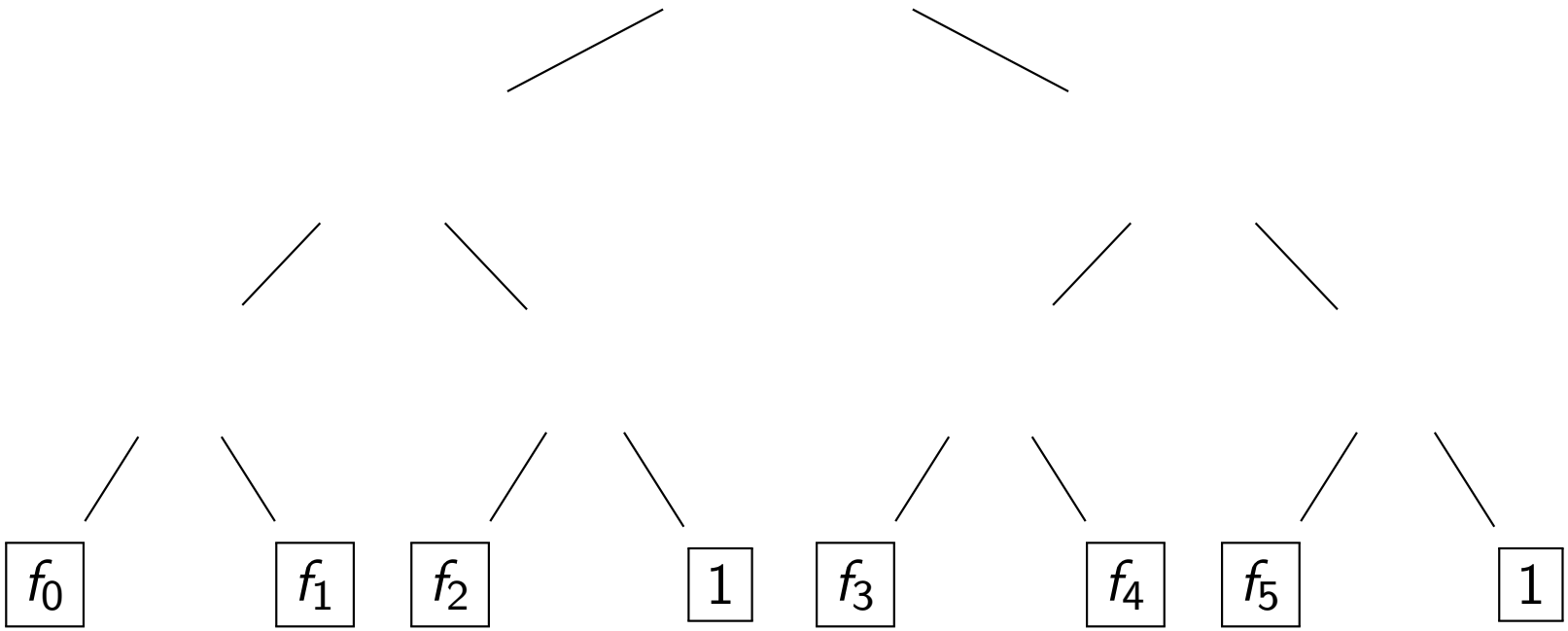
$$f_0 = Y - (-0.23 - 0.75i) \qquad f_3 = Y - (-0.23 - 0.75i)$$
$$f_1 = Y - (-0.23 + 0.75i) \qquad f_4 = Y - (-0.23 + 0.75i)$$
$$f_2 = Y - 6.9 \qquad f_5 = Y - 2.6$$

$$V^{(2)} = \boxed{Y^6 - 4Y^5 - 19Y^4 - 8Y^3 - 11Y^2 + Y - 1}$$



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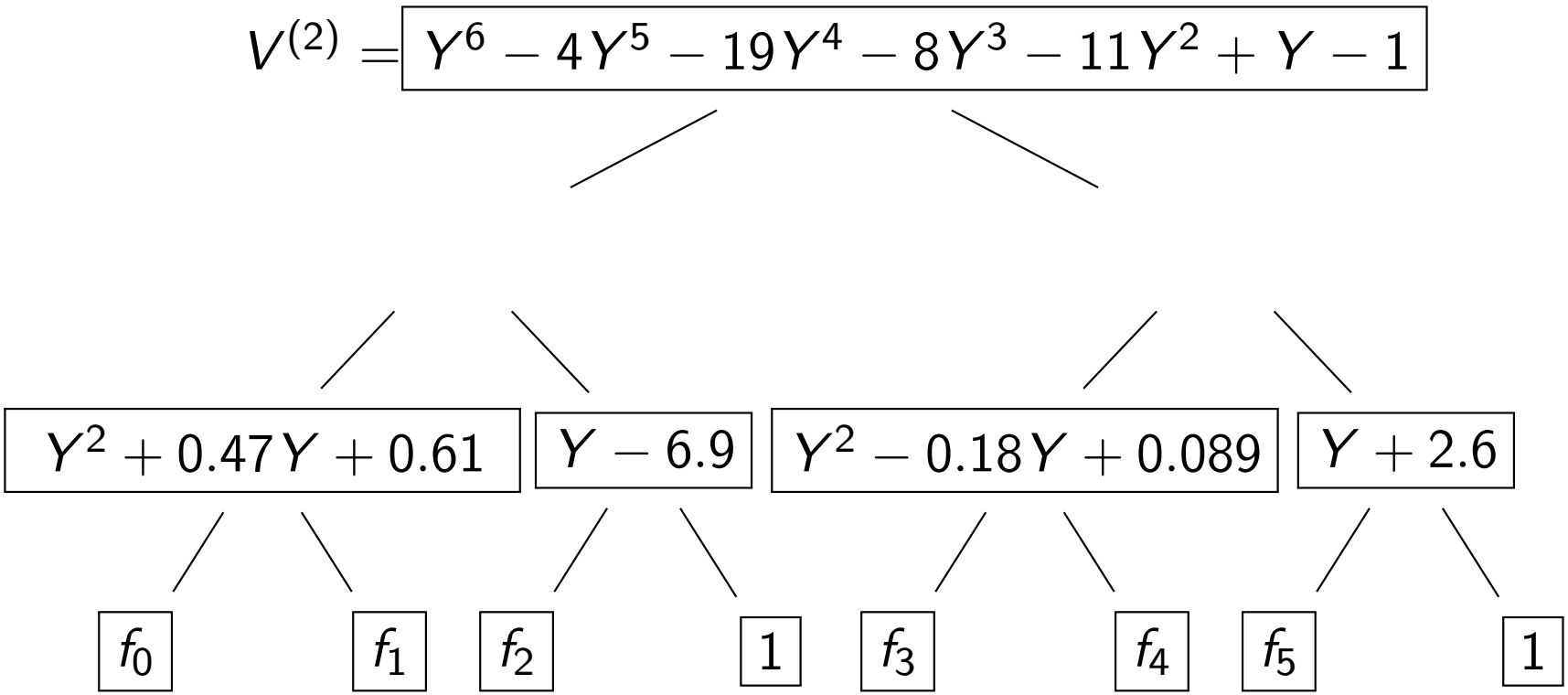

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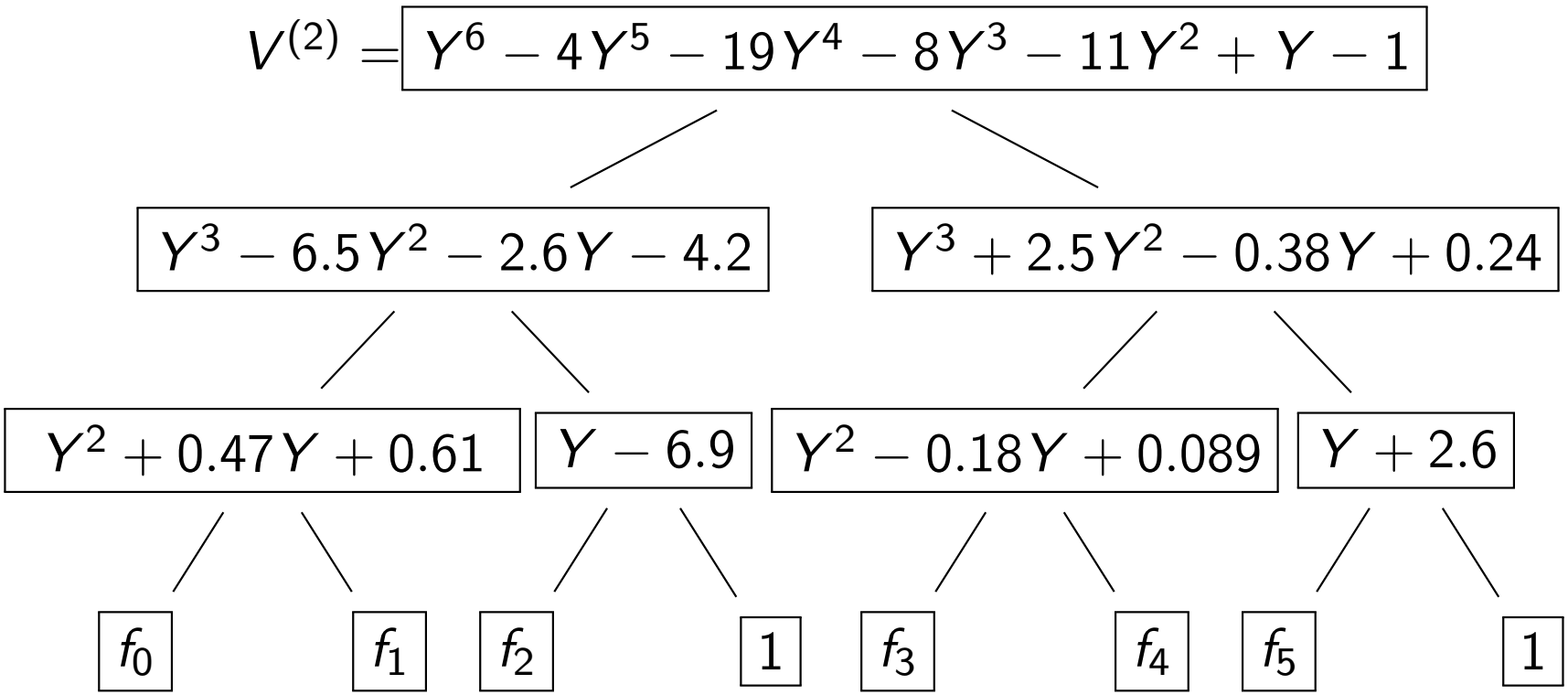

Let $M_X(n)$ be the number of multiplications in $\mathbb{C}$ required to multiply two polynomials with at most n coefficients in $\mathbb{C}[X]$. We assume that

$$M_X(n) + M_X(m) \leq M_X(n+m).$$

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Computing a subproduct tree which has s leaves takes at most

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Let $f(s)$ be the time it takes to compute a subproduct tree with s leaves. Then

$$f(s) = 2f(s/2) + M_X(s/2).$$

Moreover, $f(1) = 0$. We have

$$f(s) = \sum_{i=1}^{k} 2^i M_X(s/2) \leq \sum_{i=1}^{k} M_X(s) = kM_X(s) = O(M_X(s)\log s).$$

Algorithm

ALGORITHM

For each $\iota = 1, 2, \ldots, t-1$, *do the following:*

1. *Compute the* U_i*'s for this iteration.*
2. *Find* $V^{(\iota+1)}$ *by taking different* V_k*'s and see if they're irreducible.*
3. *Express the coefficients of* $W = U_0$ *as their Hecke representations.*

Corollary

Computing all of the U_i for one iteration takes at most

$$O(nM_X(m)\log m)$$

operations in $\mathbb{C}$.

We create a subproduct tree for each U_i.

Corollary

Computing V for one iteration takes at most

$$O(mM_X(n)\log n)$$

operations in $\mathbb{C}$.

We create a subproduct tree for each V_k and check if it is irreducible.

Algorithm

ALGORITHM

For each $\iota = 1, 2, \ldots, t-1$, do the following:

1. *Compute the U_i's for this iteration.*
2. *Find $V^{(\iota+1)}$ by taking different V_k's and see if they're irreducible.*
3. *Express the coefficients of $W = U_0$ as their Hecke representations.*

Theorem

Computing the Hecke representation of the coefficients of U_0 takes

$$O(mM_X(n)\log n)$$

operations in $\mathbb{C}$.

Theorem

One iteration of the main algorithm takes at most

$$O(mM_X(n)\log n + nM_X(m)\log m)$$

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Since we are working in $\mathbb{C}$, we can use the Fast Fourier transform where $M_X(n) = n\log n$. Under this model, we find that the main algorithm takes at most

$$O\left(mn(\log^2 m + \log^2 n)\right)$$

operations in $\mathbb{C}$.

Corollary

Let $K = \mathbb{Q}(\sqrt{-D})$ *and* $M = H_K$. *Note that the*

$$|G| = |\mathrm{Gal}(M/K)| = |\mathrm{Cl}(\mathcal{O}_K)| = h$$

where h *is the class number. Then the main algorithm requires at most*

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Assuming h is not prime, the larger integer between m and n is at most $h/2$. Hence $\log m = O(\log h)$ and $\log n = O(\log h)$. Thus,

$$O\left(mn(\log^2 m + \log^2 n)\right) = O(h \log^2 h).$$

Finally, we approximate t, the number of factors of h, to be $O(\log h)$. Note that $t - 1$ is the number of iterations we need. From there, we get the result.

Problem

Given a finite abelian extension M/K generated by the polynomial h, find an intermediate field L and the respective polynomials W and V, for M/L and L/K.

Motivation

Find a root x of the Weber polynomial $\mathscr{W}_D$.

$\Downarrow$

Find a root $j \in \mathbb{F}_p$ of the Hilbert class polynomial of $\mathbb{Q}(\sqrt{-D})$.

$\Downarrow$

Construct an elliptic curve E over $\mathbb{F}_p$ with N points where N is in terms of p and D.

$\Downarrow$

Certify primality of p using the Atkin-Morain ECPP algorithm.

We discuss the solution given in Enge (2003) more thoroughly.

Algorithm (CM algorithm)

1. *Compute the Hilbert class polynomial* $h_K \in \mathbb{Z}[X]$ *for the field* $K = \mathbb{Q}(\sqrt{-D})$ *where* $-4D = (p+1-N)^2 - 4p$.
2. *Compute a root* $j \in \mathbb{F}_p$ *of* h_K, *viewed as a polynomial over* $\mathbb{F}_p$.
3. *Let*

$$E = \begin{cases} Y^2 = X^3 + aX - a & \text{for } j \neq 0, 1728 \\ Y^2 = X^3 + 1 & \text{for } j = 0 \\ Y^2 = X^3 + X & \text{for } j = 1728 \end{cases}$$

   where $a = \frac{27j}{4(1728-j))}$.
4. *If* $|E(\mathbb{F}_p)| = N$, *return* E. *Otherwise, return its quadratic twist.*

Suppose we want to find an elliptic curve over the finite field $\mathbb{F}_p$ with N points where $p = 479$ and $N = 456$. Note that

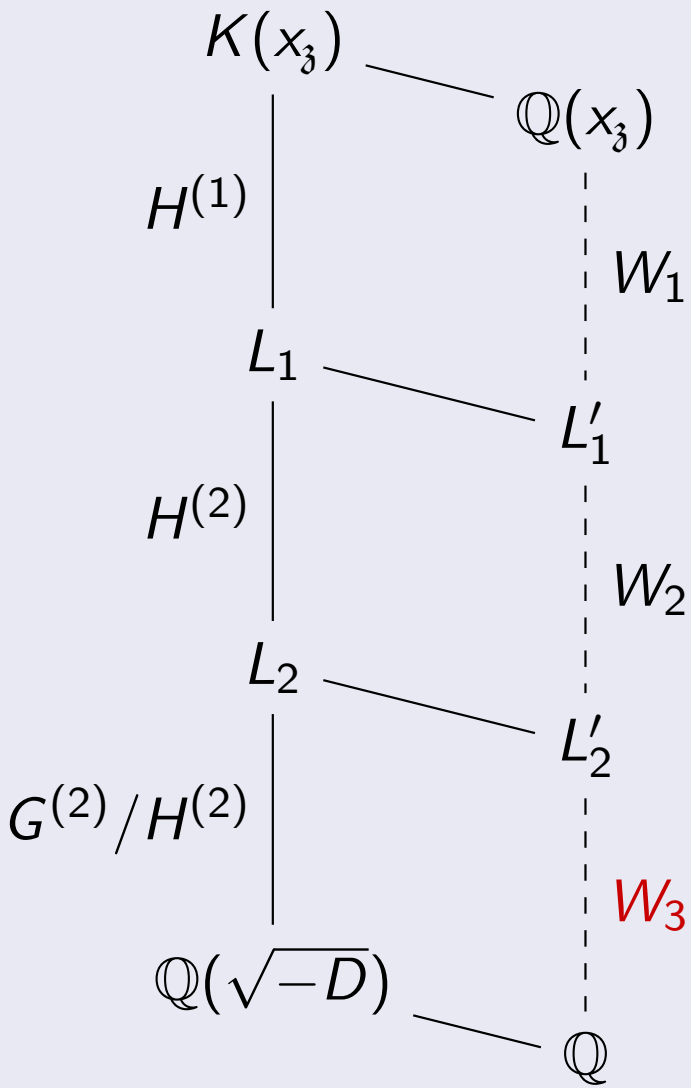
$$(p+1-N)^2 - 4p = (479+1-456)^2 - 4(479) = -1340 = -4 \cdot 335.$$

Example

$$W_1 := X^3 + \frac{1}{V^{(2)\prime}(Y)} \sum_{j=0}^{2} g_{1,j}(Y) X^j$$

$$W_2 := Y^3 + \frac{1}{V^{(3)\prime}(Z)} \sum_{j=0}^{2} g_{2,j}(Z) Y^j$$

$$W_3 := Z^2 - 4Z - 16$$



We compute in $\mathbb{F}_p$ with $p = 479$.

$$W_3(Z) = Z^2 - 4Z - 16$$

$$W_2(Y) = Y^3 + \frac{4Z + 32}{2Z - 4}Y^2 - \frac{3Z + 4}{2Z - 4}Y + \frac{4Z + 12}{2Z - 4}$$

$$W_1(X) = X^3 + \frac{g_{1,2}(Y)}{V^{(2)\prime}(Y)}X^2 + \frac{g_{1,1}(Y)}{V^{(2)\prime}(Y)}X + \frac{g_{1,0}(Y)}{V^{(2)\prime}(Y)}$$

We compute in $\mathbb{F}_p$ with $p = 479$.

$$W_3(Z) = Z^2 - 4Z - 16$$

$$W_3(46) = 46^2 - 4(46) - 16 = 0$$

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Note that for $x = 81$, we have $\mathscr{W}_{335}(x) = 0$ as a polymomial in $\mathbb{F}_p$, with $p = 479$.

We convert this into a root of the Hilbert class polynomial h_K by the formula [Konstantinou]:

$$j = \frac{(x^{-24} - 16)^3}{x^{-24}} = 117.$$

Finally, we take

$$a = \frac{27(j)}{4(1728 - j)} = 417.$$

And so we take the elliptic curve

$$Y^2 = X^3 + 417X + 62$$

and this curve has $N = 456$ points.

Problem

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Find a root x of the Weber polynomial $\mathscr{W}_D$.

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Construct an elliptic curve E over $\mathbb{F}_p$ with N points where N is in terms of p and D.

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Made an implementation to decompose towers for $D \equiv 7 \pmod 8$, $D \not\equiv 0 \pmod 3$ and $D < 1000000$.

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Took the precision $\eta = \frac{1}{9} h_D \log D$.

- Slowest was $D = 947231$ with $h_D = 883 \times 2$, which took 5.382 seconds.
- Slowest "smooth" class number was $D = 905339$ with $h_D = 1536 = 2^9 \times 3$, which took 2.087 seconds.

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Took the precision $\eta = \frac{1}{9} h_D \log D$.

- Slowest was $D = 947231$ with $h_D = 883 \times 2$, which took 5.382 seconds.
- Slowest "smooth" class number was $D = 905339$ with $h_D = 1536 = 2^9 \times 3$, which took 2.087 seconds.

Only 5 instances where taking the trace does not result in an irreducible V.

Additional observations

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- We do not actually need to compute the Weber polynomial.
- We can lower the precision after each iteration since the coefficients will in general be of lower magnitude.

Tower decomposition of Hilbert class fields

Jared Guissmo E. Asuncion

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